OSU Department of Mathematics
Qualifying Examination
Fall 2024

Real Analysis

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notations:

- $C^1[0,1]$ denotes the space of all real-valued continuously differentiable functions on $[0,1]$.
- In the context of metric spaces, the notation (M, d_M) is used for a metric space M with metric d_M.
- For any pair of topological spaces X and Y, we let $X \times Y$ denote the product space equipped with the product space topology.

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Problems:

1. Recall that a real-valued function φ is *convex* on $[0,\infty)$ if

$$\varphi\big((1-\lambda)x+\lambda y\big) \le (1-\lambda)\varphi(x)+\lambda\varphi(y) \qquad \forall x,y\in[0,\infty) \;\text{ and }\; \forall\lambda\in[0,1].$$

Let $\mathcal{W}=\Big\{w=\big(w_i\big)_{i=0,1,2,\dots} : w_i\ge 0 \;\; \forall i=0,1,2,\dots \;\text{ and }\; \sum_{i=0}^{\infty} w_i = 1\Big\}$ denote the space of all weight functions indexed by nonnegative integers and consider the following inequality: for all $w\in\mathcal{W}$ and all real valued convex functions $\varphi:[0,\infty)\to\mathbb{R}$,

$$\sum_{i=0}^{\infty}\varphi(i)\,w_i \ge \varphi\Big(\sum_{i=0}^{\infty} i\,w_i\Big) \tag{1}$$

whenever both series converge. Prove the following variants of (1).

a. (2 pts) Show that inequality (1) holds for all $w\in\mathcal{W}$ such that $\sum_{i=0}^{1} w_i = 1$.

b. (5 pts) For all integer $n\ge 1$, use induction to prove that inequality (1) holds for all $w\in\mathcal{W}$ such that $\sum_{i=0}^{n} w_i = 1$.

c. (3 pts) Use part **b** to prove (1) when φ is assumed to be continuous on $[0,\infty)$.

2. For all $n\in\mathbb{N}$, let $f_n:[0,\infty)\to\mathbb{R}$ be defined by $f_n(x)=\sin\sqrt{x+4n^2\pi^2}$. Prove that:

a. (1 pt) For all $x\in[0,\infty)$, $f_n(x)\to 0$ as $n\to\infty$ (pointwise convergence).

b. (4 pts) The collection $\{f_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous.

c. (3 pts) There is no subsequence of $\{f_n\}$ which converges uniformly to the zero function.

d. (2 pts) State the Arzela-Ascoli Theorem and explain why the example in this problem is not a counterexample to this theorem.

3. (10 pts) Consider a sequence $f_n(x)$ of real valued functions in $C^1[0,1]$ such that

$$|f_n(x)|+|f_n'(x)| \le -\frac{\ln x}{\sqrt{x}} \qquad \text{for all } x\in(0,1] \text{ and all } n.$$

Prove that the sequence $f_n(x)$ has a uniformly converging subsequence.

4. **a.** (4 pts) Let (X,d_X) be a metric space and $K\subseteq X$ be compact. Prove that K is closed.

b. (6 pts) Let (X,d_X) and (Y,d_Y) be metric spaces and let $f:X\to Y$ be a map. Suppose that the graph of f,

$$G_f := \{(x,f(x))\in X\times Y \,|\, x\in X\},$$

is compact. Prove that f is continuous. Hint: Recall that for metric spaces, being compact is the same as being sequentially compact.