Stochastic Explosion and Non-Uniqueness for α-Riccati Equation.

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Abstract

We consider the problem of global in time existence and uniqueness for the initial value problem $u'(t) = -u(t) + u^2(\alpha t), u(0) = u_0 \geq 0$ where $\alpha \geq 0$ is a fixed parameter. We focus on the "super-critical" case $\alpha > 0$ where by using the multiplicative stochastic cascade techniques we prove global existence for small initial data and finite-time blow-up large initial data. However, while uniqueness holds for solutions satisfying a growth condition, in general it fails even for arbitrary small initial data. We demonstrate that this lack of uniqueness is directly connected to the stochastic explosion in the associated multiplicative stochastic cascade process. The key tool in establishing the above-mentioned results is an iterative algorithm that allows one to exploit the stochastic explosion of the underlying multiplicative cascade to establish both existence and lack of uniqueness of solutions.

1 Introduction

We are interested in the existence and uniqueness of global in time nonnegative solutions to the following equation, which we will refer to as α-Riccati, equation

$$u'(t) = -u(t) + u^2(\alpha t), \quad u(0) = u_0 \geq 0, \tag{1.1}$$

for various values of the parameter $\alpha \geq 0$. Note that in the case $\alpha = 1$ (1.1) becomes a classical Riccati-type equation (also known as logistic equation). We retain the name *Riccati* for (1.1) to stress its quadratic nonlinearity that, as we will see later, leads to a finite-time blow-up for large initial data in the case in the case $\alpha \geq 1$ (cf. Section 5). The uniqueness problem for this system had originally been considered, somewhat indirectly, in [1] and, in the case $u_0 = 0$[1], in [2]. The case $\alpha \leq 1$ was considered in [7] and in [6].

Our interest in (1.1) is motivated by the program to further develop stochastic cascade methods to analyze the well-posedness problems pioneered by Le Jan and Sznitman in [14]. We modify their techniques to permit stochastically exploding cascades, and thereby obtain new perspectives on existence and uniqueness problems. In particular, this approach may provide provide an alternative pathway to tackle global existence and uniqueness problems for the 3D Navier-Stokes and related equations. In its original formulation the Le Jan-Sznitman cascade yielded global existence and uniqueness for small initial data results. It was subsequently adapted in [6] to establish existence and uniqueness for arbitrary large initial data in the case of complex Burgers and its simplified β-field versions, which are exactly α-Riccati equation with

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[1] In fact, the author of [2] analyzed a more general form of the problem that allowed for mild forms of the equation with general branching and holding time distributions.

$\alpha = \beta^2 \in [0,1]$. The connection between uniqueness, scaling symmetry and *non-explosion* of associated stochastic cascades was considered in [5][2] for the 3D Navier-Stokes equations.

The above-mentioned PDE systems possess the same basic structure if viewed as a mild formulation in Fourier space. It consists of a multiplicative linear part and a convolution-like non-linearity. The system (1.1) has a similar, albeit much simpler structure; the parameter α can be viewed as modeling Fourier-space interactions for scaling-invariant solutions. It plays a similar role to β in the β-field Burgers equation considered in [6].

More precisely, many nonlinear PDEs, including the 3D incompressible Navier-Stokes equations (NSE), have the following form when viewed in an appropriate Hilbert space settings (see e.g.[15])

$$u'(t) = -Au + B(u,u), \tag{1.2}$$

where A is an unbounded positive linear operator (usually a Fourier multiplier) and $B(u,u)$ is a non-linearity of quadratic type. Using the natural scaling of the equation, it may be possible to re-write (1.2) as a mild formulation in Fourier space with respect to similarity variables, which, in the NSE case, are $\tau = |\xi|^2 t$ and $e_\xi = \xi/|\xi|$:

$$v(\tau, e_\xi) = v_0(e_\xi)e^{-\tau} + \int_0^\tau e^{-\sigma} \int_{\mathbb{R}^n} v(|\eta|^2(\tau-\sigma), e_\eta) \odot_{e_\xi} v(|\eta - e_\xi|^2(\tau-\sigma), e_{e_\xi - \eta}) H(\eta|e_\xi), \mathrm{d}\eta\, \mathrm{d}\sigma\,, \tag{1.3}$$

where $\odot_{e_\xi}$ is a product structure reflecting the geometry on the nonlinearity B, and the kernel $H(\eta|\xi)$ arises from the natural scaling symmetry of the equation and is normalized so that $\int_{\mathbb{R}^n} H(\eta|e_\xi)\,\mathrm{d}\eta = 1$ for any ξ. In the NSE case $H(\eta|e_\xi) = c/(|\eta|^2|e_\xi - \eta|^2)$ (see [5]). In the case of the complex Burgers equation, $H(\eta|e) = \mathbb{1}_{[0,1]}$ (see [6]).

The key observation going back to [14] is that a solution to (1.3) represents an expected value $v(\tau, e_\xi) = \mathbb{E}(X(\tau, e_\xi))$ for a certain stochastic process $X(\tau, e_\xi)$, which satisfies $X(0, e_\xi) = v_0(e_\xi)$ and is associated with a certain multiplicative cascade such that the right-hand side of the equation represents the conditioning on the first branching (see [5] for a rigorous description of this *self-similar* cascade for the NSE, and [6] for the complex Burgers case). It is interesting to note that due to the rotational symmetry of the probability kernel $H(\cdot|e_\xi)$, if $v_0(\cdot)$ is rotation-invariant, then so is $X(\tau, \cdot)$ (in distribution sense).

Thus, in the rotation-invariant case $X(\tau, e_\xi) = X(\tau)$, and if we ignore the geometry given by $\odot$-product and simply replace it with the product of magnitudes, and further simplify by replacing $H(\cdot|\cdot)$ with the Dirac distribution $\delta_{\sqrt{\alpha}}$ with $\alpha \geq 0$ – magnitude-squared of η, and finally ignore the difference between $|\eta - e_\xi|$ and $|\eta|$ (natural if $\alpha \gg 1$), then (1.3) simplifies into

$$v(\tau) = v_0 e^{-\tau} + \int_0^\tau e^{-\sigma}\, v^2(\alpha(\tau - \sigma))\,\mathrm{d}\sigma,$$

which is *precisely* the variation of constants formulation for the α-Riccati problem (1.1).

Note that the case when $H(\cdot|e_\xi)$ tends to generate larger frequencies roughly corresponds to the case $\alpha > 1$, while the case where H has a high chance of producing small frequencies corresponds to $\alpha < 1$. Conceptually, as we will see later, larger frequencies lead to faster branching clocks in the associated Le Jan-Sznitman cascade, resulting in a denser tree structure and possibly stochastic explosion – which fundamentally changes the nature of the equation.

Thus our aim is to investigate the use stochastic explosion in the Le Jan-Sznitman cascades to show lack of uniqueness as well as finite-time blow up in (1.1). In particular, we are building on [7] and [2] to establish

[2][CORRECTION]: Although the proofs remain unchanged, the non-explosion statement in Propositions 5.1 and 5.3 of [5] should read $P(\zeta < \infty) = 0$ in place of $P(\zeta = \infty) = 0$.

an explicit perspective on how the *uniqueness/non-uniqueness* phenomena for (1.1) can be fully expressed in terms of *non-explosion/explosion* of an associated branching random walk.

The relationship between stochastic explosion and uniqueness of the solutions to the associated Cauchy problem for the (linear) Kolmogorov backward equations for Markov processes is well-known. In particular, explosion permits the Markovian return of the process to the state space upon explosion in a manner that preserves the local rules of evolution captured by the infinitesimal generator, while distinguishing the global character of the process and its transition probabilities; see ([9], pp. 488-491), ([3], pp. 612-616). On the other hand, while uniqueness criteria for certain semi-linear PDEs have been known to have analytical formulations in terms of explosion of branching processes, see [11], [10], the stochastic mechanisms have been less transparent. A connection between stochastic non-explosion and uniqueness of solutions of non-linear PDE was conjectured in [5] for the 3D Navier-Stokes equations. This conjecture was based on the observation that non-explosion of the underlying stochastic cascade seems to be instrumental in proving uniqueness.

Remark 1.1. The infinitesimal generator governing the genealogical evolution of the associated stochastic cascade for the α-Riccati equation is given in [7]. The corresponding backward equations may be used to obtain expected values of various functionals associated with the cascade.

To our knowledge this paper provides the first explicit example and a general strategy of how to use stochastic explosion to build non-unique solutions for nonlinear differential equations for large classes of initial data. The key tool for our analysis is an iterative procedure outlined in Section 4 which, in the case of stochastic explosion becomes a mechanism for generating multiple solutions for fixed initial data. This method allows to establish explicit connection between stochastic explosion and the non-uniqueness problem for (1.1).

In the case $\alpha \in [0, 1]$ – see [7] – the underlying stochastic process, called *a delayed Yule Process*, is non-exploding. Since a solution to (1.1) can be constructed using the moment generating function for the number of branches of the process by time t, the existing results imply, for $\alpha \in [0, 1)$, both existence and uniqueness of global solutions for arbitrary large initial data. In the case $\alpha = 1$, as it readily follows from the explicit calculations, the global solutions exist and are unique if and only if $u_0 \in [0, 1]$, with finite-time blow-up for $u_0 > 1$. In all of the above cases the non-explosive property of the associated (delayed Yule) cascades was crucial to establish uniqueness of solutions in this approach (cf. Section 5). Note that the two terms "stochastic explosion"and "blow-up"have completely different technical meanings.

This paper mainly focuses on the remaining case: $\alpha > 1$, which leads to stochastically exploding cascades. Among the main results, we show that 1) the global solutions exist for small enough initial data (surprisingly, the bound of the data grows to infinity with α); 2) finite time blow up for large initial data; 3) Non-uniqueness of global solutions, including for arbitrary small initial data (uniqueness can be achieved by suitably restricting behavior at infinity); 4) local in time existence and non-uniqueness for arbitrary initial data. In addition, we illustrate the use of the stochastic cascade to analyze long-time behavior of the global solutions (see Section 6), and establish a duality, in probabilistic sense, between biggest and smallest solutions for $u_0 = 0$ and $u_0 = 1$ respectively, which points to deeper connections between underlying probabilistic structures and the issue of uniqueness (Section 8). Moreover, we further extend applicability of our approach to non-uniqueness based on the stochastic explosion in the Le Jan-Sznitman cascades by recovering a remarkable solution of α-Riccati equations obtained by Athreya in [2] (see Section 9). Finally, we compare the results we obtained using the probabilistic approach with what can be gained from direct estimates, showing the remarkable consistency between the two seemingly unrelated approaches (Section 5).

The main takeaway of our analysis is a broad unification of the uniqueness theory in terms of two stochastic phenomena: (i) explosion, and (ii) a special duality relationship derived from the initial data.

We also mention that very recently we were able to show explosion for the self-similar stochastic cascade associated with the 3D Navier-Stokes equations (see [8]). Thus, the current paper can serve as an indication that the self-similar solutions to these equations may not be unique. This would be consistent with the recent results on the uniqueness of the Navier-Stokes problem ([13], [12], [4]).

The paper is organized as follows. In Section 2 we introduce the relevant notation and terminology and recall basic results from previous work. Section 3 contains main existence and (conditional) uniqueness results based on the stochastic cascade approach. In Section 4 we introduce a system of iterative stochastic processes and use it to build two classes of solutions based on the hyper-explosion property of the stochastic cascade, leading to non-uniqueness. Section 5 deals with finite-time blow-up for large initial data, both from the stochastic cascades and from direct analysis perspectives. Section 6 contains several results related to the asymptotic behavior of global solutions. Section 7 deals with existence and the lack of uniqueness of local in time solutions. Finally, Section 8 deals with duality of between "minimal" and "maximal" solutions, and Section 9 provides another application of our iterative technique by recovering a remarkable solution obtained in [2].

2 Preliminaries: Le Jan-Sznitman Cascade.

Recall that the α-Riccati equation was defined as

$$\left\| \begin{array}{l} u'(t) = -u(t) + u^2(\alpha t) \\ u(0) = u_0 \geq 0 \end{array} \right. . \tag{2.1}$$

In integral form the above system becomes

$$u(t) = u_0\, e^{-t} + \int_0^t e^{-s}\, u^2(\alpha(t-s))\,\mathrm{d}s = u_0\, e^{-t} + e^{-t}\int_0^t e^{s}\, u^2(\alpha\, s)\,\mathrm{d}s \qquad u_0 \geq 0\,. \tag{2.2}$$

We associate to this equation the usual *Le Jan-Sznitmann multiplicative stochastic cascade (branching random walk)*, consisting of a binary tree indexed by $v \in \mathcal{I}$ where

$$\mathcal{I} = \{\emptyset\} \cup \left(\bigcup_{n\in\mathbb{N}} \{1,2\}^n \right) .$$

(with root labeled by $\emptyset$) and decorated by the iid exponential mean one random variables T_v.

Given a branch with genealogy index $v \in \mathcal{I}$ we define the random variable measuring the (time-)length of this branch

$$\theta_v = \sum_{n=0}^{|v|} \frac{T_{v|n}}{\alpha^n}, \tag{2.3}$$

where $|v|$ is the length – number of terms (*generations*) of a finite sequence (branch) v, and $v|n$ represents the first n terms of the sequence v, with the convention $|\emptyset| = 0$, and $v|0 = \emptyset$.

Also consider the random variables representing the length of the shortest path (also called *explosion time*):

$$S = \inf_{w\in\{1,2\}^{\mathbb{N}}} \sum_{k=0}^{\infty} \frac{T_{w|k}}{\alpha^k} = \lim_{n\to\infty} \min_{|v|=n} \sum_{k=0}^{n} \frac{T_{v|k}}{\alpha^k}$$

and the length of the longest path:

$$L = \sup_{w\in\{1,2\}^{\mathbb{N}}} \sum_{k=0}^{\infty} \frac{T_{w|k}}{\alpha^k} = \lim_{n\to\infty} \max_{|v|=n} \sum_{k=0}^{n} \frac{T_{v|k}}{\alpha^k}$$

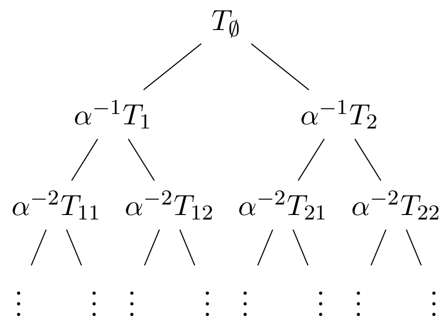

Figure 1: Le Jan-Sznitman cascade; $\theta_{21} = T_\emptyset + \alpha^1 T_2 + \alpha^{-2} T_{21}$

We associate to S an L the *explosion event* $\mathrm{E} = \{\mathrm{S} < \infty\}$ and the *hyper-explosion event* $\mathrm{F} = \{\mathrm{L} < \infty\}$. Clearly, $\mathrm{F} \subseteq \mathrm{E}$. Also, if $S < t$, then the multiplicative cascade generates infinitely many branches by time t.

Given t we say that a branch $v \in \mathcal{I}$ *crossed* t in if it its time-length exceeds t but the time-length of the immediate predecessor does not:

$$\sum_{k=0}^{|v|} \frac{T_{v|k}}{\alpha^k} \geq t \quad \text{and} \quad \sum_{k=0}^{|v|-1} \frac{T_{v|k}}{\alpha^k} < t.$$

We also say that a branch v has *survived by time* t if its time-length $\theta_v < t$.

Our construction of solutions of (2.2) will depend in the essential way on whether the underlying Le Jan-Sznitman cascade can develop infinitely many branches in finite time – a phenomenon called *stochastic explosion*. In particular, the stochastic cascade is called non-exploding if

$$S = \infty \text{ a.s.} \quad (\text{Alternatively, } \mathbb{P}(\mathrm{E}) = 0.)$$

exploding if

$$S < \infty \text{ a.s.} \quad (\mathbb{P}(\mathrm{E}) = 1.)$$

and *hyper-exploding* if

$$L < \infty \text{ a.s.} \quad (\mathbb{P}(\mathrm{F}) = 1.)$$

The main observation is that α-Riccati cascade is *hyper-exploding* for $\alpha > 1$.

Theorem 2.1. *Consider the Le Jan-Sznitman cascade for $\alpha \geq 0$.*

1. *If $0 \leq \alpha \leq 1$ then $L = S = \infty$ a.s.*
2. *If $\alpha > 1$ then both $S < \infty$ and $L < \infty$ a.s. In fact,*

$$\lim_{t\to\infty} \mathbb{P}(S > t) = 0, \tag{2.4}$$

$$\lim_{t\to\infty} \mathbb{P}(L \leq t) = 1, \tag{2.5}$$

and thus $\mathbb{P}(\mathrm{E}) = \mathbb{P}(\mathrm{F}) = 1$.

Proof. The proof of part 1 was given in [7]. In fact, for any path $v \in \{1,2\}^{\infty}$, with probability one,

$$\sum_{j=0}^{\infty} \alpha^{-j} T_{v|j} \geq \sum_{j=0}^{\infty} T_{v|j} = \infty,$$

by, for example, the strong law of large numbers.

For part 2, the a.s. finiteness of L was proven in [2] in the case $\alpha > 1$. Thus $S \leq L < \infty$ a.s. as well. The argument in [2] was to note that the sequence $L_n = \max_{|v|=n} \sum_{j=0}^{n} \alpha^{-j} T_{v|j}, n \geq 1$, may be bounded iteratively by

$$L_{n+1} \leq L_n + \Theta_{n+1} \leq T_{\emptyset} + \sum_{n=1}^{\infty} \Theta_n,$$

where $\Theta_n = \alpha^{-n} \max\{T_n^{(1)}, \ldots, T_n^{(2^n)}\}$, where $T_n^{(j)}$ are i.i.d. mean one exponential random variables. Thus, for $L = \lim_{n\to\infty} L_n \leq T_{\emptyset} + \sum_{n=1}^{\infty} \Theta_n$, it sufficient to show $\sum_{n=1}^{\infty} \Theta_n < \infty$ a.s. Fix a sequence $\theta_n, n \geq 1$, to be determined and consider

$$\begin{aligned}
\mathbb{P}(\Theta_n > \theta_n) &= 1 - \mathbb{P}(\Theta_n \leq \theta_n) \\
&= 1 - (1 - e^{-\theta_n \alpha^n})^{2^n} \\
&\leq e^{n \ln 2 - \theta_n \alpha^n} = e^{-n},
\end{aligned} \tag{2.6}$$

for $\theta_n = n(\ln 2 + 1)\alpha^{-n}$. Thus, using Borel-Cantelli lemma, one has with probability one $\Theta_n \leq \theta_n$ for all but finitely many n, and therefore $\sum_{n=1}^{\infty} \Theta_n < \infty$ a.s. Thus, the finiteness of L as well as (2.4) and (2.5) follow. □

Remark 2.1. An immediate connection between the Le Jan-Sznitman cascade described above and (2.2) is established by noting that $\underline{u}_1(t) = \mathbb{P}(S > t)$ – the complementary distribution function of the length of the shortest branch, and $\overline{u}_0(t) = \mathbb{P}(L \leq t)$ – the distribution function of the length of the longest branch, are both solutions to the equation. This fact could be verified by computing the above probabilities via conditioning on the first branching of the Le Jan-Sznitman tree. The solution $\overline{u}_0(t)$ and its asymptotic behavior as $t \to \infty$ was first studied in [2]. In the case $\alpha \in [0,1]$, $\underline{u}_1(t) = 1$ and $\overline{u}_0(t) = 0$ – the constant solutions of (2.2). In Section 8 we establish a duality relation between these two solutions.

The next propositions refines the manner in which the Le Jan-Snitman tree for (2.2) may explode.

Proposition 2.1. *Consider the event*

$$\mathrm{G_t} = \{\textit{zero or finitely many branches crossed } t.\}$$

Then for any any $t \geq 0$,

$$\mathbb{P}(\mathrm{G_t}) = 1. \tag{2.7}$$

Proof. First note that $u(t) = \mathbb{P}(\mathrm{G_t})$ is the solution of (2.2) with $u(0) = 1$. Just like above this can be established by conditioning on the first branching. Also, since $\{L \leq t\} \subseteq \mathrm{G_t}$, by Theorem 2.1,

$$\lim_{n\to\infty} u(t) = 1.$$

Denote $q(t) = 1 - u(t)$. Observe that q satisfies $0 \leq q(t) \leq 1$, $\lim_{t\to\infty} q(t) = 0 = q(0)$, and

$$q'(t) = -q(t) + 2q(\alpha t) - q^2(\alpha t). \tag{2.8}$$

Claim 1.

$$\int_0^\infty q(s)\,\mathrm{d}s < \infty. \tag{2.9}$$

Proof. First observe that $0 \le q(t) \le 1 - \mathbb{P}(L \le t)$. For $\alpha \ge 2$, by the asymptotic estimates in [2] we have $1 - \mathbb{P}(L \le t)$ is integrable on $[0, \infty)$, and (2.9) holds.

For $\alpha < 2$, integrating (2.8) on $[0, t]$ yields

$$q(t) = -\int_0^t q(s)\,\mathrm{d}s + \frac{2}{\alpha}\int_0^{\alpha t} q(s)\,\mathrm{d}s - \frac{1}{\alpha}\int_0^{\alpha t} q^2(s)\,\mathrm{d}s.$$

Thus

$$\alpha q(t) = \int_0^t ((2-\alpha)q - q^2) + \int_t^{\alpha t} (2q - q^2). \tag{2.10}$$

If by contradiction we assume $\int_0^\infty q = \infty$, since $q(t) \to 0$ as $t \to \infty$, we see that

$$\int_0^\infty ((2-\alpha)q - q^2) = \infty \quad \text{and} \quad \int_t^{\alpha t} (2q - q^2) \ge 0,$$

Therefore, taking $t \to \infty$ in (2.10) we obtain a contradiction, which yields (2.9) for $\alpha < 2$. □

Returning to the proof of Proposition, use variation of constants on (2.8) to obtain

$$q(t) = e^{-t}\int_0^t e^s (2q - q^2)(\alpha s)\,\mathrm{d}s \ge \int_0^t (2q - q^2)(\alpha s)\,\mathrm{d}s.$$

Letting $t \to \infty$, we obtain

$$\int_0^\infty (2q - q^2) = 0.$$

But since $0 \le q \le 1$, by (2.9) we also have

$$\int_0^\infty q^2 \le \int_0^\infty q < \infty.$$

Consequently, $q(t) = 0$ for all $t \ge 0$ which implies (2.7).

□

Although in case $\alpha > 1$, by Theorem 2.1, the cascade is hyper-exploding, the chance of the non-exploding tree by finite time $t > 0$, $\mathbb{P}(S > t)$, is non-zero. Define

$$P_n(t) = \mathbb{P}(S > t \text{ and exactly } n \text{ branches crossed time } t, S \ge t). \tag{2.11}$$

Clearly, $\mathbb{P}(S > t) = \sum_{n=1}^{\infty} P_n(t)$. Note that

$$P_1(t) = e^{-t},$$

and, conditioning on the first branching, for $n > 1$:

$$P_n(t) = \int_0^t e^{-s} \sum_{k=1}^{n-1} P_k(\alpha(t-s))\, P_{n-k}(\alpha(t-s))\, \mathrm{d}s = e^{-t} \int_0^t e^{\sigma} \sum_{k=1}^{n-1} P_k(\alpha\sigma)\, P_{n-k}(\alpha\sigma)\, \mathrm{d}\sigma.$$

By induction, we can show the following estimates on P_n.

Lemma 2.1. *Assume $\alpha > 1$. Then for any $n \in \mathbb{N}$:*

$$\frac{C_n}{(2\alpha-1)^{n-1}}\, e^{-t} \geq P_n(t) \geq \frac{1}{(2\alpha-1)^{n-1}}\, e^{-t} \left(1 - e^{-(2\alpha-1)t}\right)^{n-1}, \tag{2.12}$$

where C_n are the Catalan numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}} \quad \textit{(as } n \to \infty\textit{)}.$$

Proof. We use induction in n. Note that Catalan numbers satisfy the recursive relation

$$C_1 = 1, \quad C_n = \sum_{k=1}^{n-1} C_k C_{n-k}.$$

Clearly, (2.12) is satisfied for $n = 1$.

Assume (2.12) holds for $k \leq n$. Then

$$P_{n+1}(t) = e^{-t} \int_0^t e^{\sigma} \sum_{k=1}^{n} P_k(\alpha\sigma)\, P_{n+1-k}(\alpha\sigma)\, \mathrm{d}\sigma \leq e^{-t} \int_0^t e^{\sigma} \sum_{k=1}^{n} \frac{C_k}{(2\alpha-1)^{k-1}} e^{-\alpha\sigma}\, \frac{C_{n+1-k}}{(2\alpha-1)^{n-k}} e^{-\alpha\sigma}\, \mathrm{d}\sigma$$

$$\leq \frac{\sum_{k=1}^{n} C_k C_{n+1-k}}{(2\alpha-1)^{n-1}}\, e^{-t} \int_0^t e^{-(2\alpha-1)\sigma} \mathrm{d}\sigma = \frac{C_{n+1}}{(2\alpha-1)^n}\, e^{-t} \left(1 - e^{-(2\alpha-1)t}\right) \leq \frac{C_{n+1}}{(2\alpha-1)^n}\, e^{-t},$$

and so the first inequality in (2.12) holds for any $n \in \mathbb{N}$.

For the lower bound we have

$$\begin{aligned} P_{n+1}(t) &= e^{-t} \int_0^t e^{\sigma} \sum_{k=1}^{n} P_k(\alpha\sigma)\, P_{n+1-k}(\alpha\sigma)\, \mathrm{d}\sigma \\ &\geq e^{-t} \int_0^t e^{\sigma} \sum_{k=1}^{n} \frac{e^{-\alpha\sigma}}{(2\alpha-1)^{k-1}} \left(1 - e^{-(2\alpha-1)\alpha\sigma}\right)^{k-1} \frac{e^{-\alpha\sigma}}{(2\alpha-1)^{n-k}} \left(1 - e^{-(2\alpha-1)\alpha\sigma}\right)^{n-k} \mathrm{d}\sigma \\ &= \frac{n}{(2\alpha-1)^{n-1}}\, e^{-t} \int_0^t e^{-(2\alpha-1)\sigma} \left(1 - e^{-(2\alpha-1)\alpha\, t}\right)^{n-1} \mathrm{d}\sigma = \frac{e^{-t}}{(2\alpha-1)^n} \left(1 - e^{-(2\alpha-1)t}\right)^n, \end{aligned}$$

and thus the second inequality in (2.12) holds as well. □

3 Minimal Solution – global existence and uniqueness results for small initial data.

In this section we consider the existence of *global in time solutions*, i.e. solutions to (2.2) that are defined for all $t \geq 0$. As noted in the introduction, the motivation coming from the Navier-Stokes-type systems makes the theory of global solutions (as opposed to local solutions considered in Section 7) especially relevant to our investigations. The importance of global solutions is especially apparent in the case $\alpha > 1$, as the equation become highly non-local (the local change of $u(t)$ depending on a later time αt).

Consider the stochastic process

$$\underline{X}(t) = u_0^{N(t)}\, \mathbb{1}_{S \geq t}, \tag{3.1}$$

where $N(t)$ is the number of branches that crossed t.

Note that $\underline{X}(t)$ satisfies

$$\underline{X}(t) = \begin{cases} 0, & S < t \\ u_0, & T_\emptyset \geq t \\ \underline{X}^{(1)}(\alpha(t - T_\emptyset))\, \underline{X}^{(2)}(\alpha(t - T_\emptyset)), & T_\emptyset < t \text{ (and } S \geq t) \end{cases} \tag{3.2}$$

where $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are iid $\underline{X}$.

The connections between $\underline{X}$ and solutions to (2.2) are summarized in the following theorems.

Theorem 3.1. *For any* $u_0 \geq 0$

- *If* $\mathbb{E}(\underline{X}(t)) < \infty$ *for* $t \geq 0$*, then* $\underline{u}(t) = \mathbb{E}(\underline{X}(t))$ *is a solution to (2.2).*
- *If* $u(t)$ *is a solution to (2.2), then* $u(t) \geq \underline{u}(t)$.

Proof. The fact that $\underline{u}$, when it is finite, is a solution of (2.2) follows form (3.2) when we compute $\mathbb{E}(\underline{X})$ by conditioning on the first branching of the stochastic tree.

To prove minimality of $\underline{u}$, assume $u(t)$ is a global solution to (2.2). Define the following sequences of stochastic processes

$$X_0\,(t) = 0, \qquad X_n(t) = \begin{cases} u_0, & T_\emptyset \geq t \\ X_{n-1}^{(1)}(\alpha(t - T_\emptyset))\, X_{n-1}^{(2)}(\alpha(t - T_\emptyset)), & T_\emptyset < t \end{cases}, \qquad n \in \mathbb{N}, \tag{3.3}$$

and

$$Y_0\,(t) = u(t), \qquad Y_n(t) = \begin{cases} u_0, & T_\emptyset \geq t \\ Y_{n-1}^{(1)}(\alpha(t - T_\emptyset))\, Y_{n-1}^{(2)}(\alpha(t - T_\emptyset)), & T_\emptyset < t \end{cases}, \qquad n \in \mathbb{N}, \tag{3.4}$$

In the above, $X_n^{(1)}$ and $X_n^{(2)}$ are iid X_n, same for Y. More explicitly,

$$X_n(t) = u_0^{N_n(t)} 0^{M_n(t)},$$

where $N_n(t)$ is the number of branches v with $|v| < n$ that cross t and $M_n(t)$ is the number of branches of length $|v| = n$ that survive by time t. Also,

$$Y_n(t) = u_0^{N_n(t)} \prod_{\substack{|v|=n, \\ v \text{ survives by } t}} u(\tau_v)\ ,$$

where $\tau_v = \alpha\,(\tau_{v|k} - T_{v|k})$ with $k = |v| - 1$ and $\tau_\emptyset = t$.

Clearly, $X_n(t) \le Y_n(t)$ a.s. Moreover, since $X_n(t)$ is eventually monotone (constant) in n if $S \ge t$ and $X_n(t) = 0$ in $S < t$, we see that

$$\lim_{n\to\infty} X_n(t) = \underline{X}(t).$$

Also, using the induction in n, $\mathbb{E}(Y_n(t)) = u(t)$ for all $n \in \mathbb{N}$. Thus, by Fatou's lemma, $\mathbb{E}(\underline{X}(t)) \le u(t)$, which proves the second statement of the theorem.

□

We will consequently refer to $\underline{X}$ and $\underline{u}$ as the *minimal solution process* and *minimal solution* to (2.2) respectively.

One can prove the following existence and uniqueness result for small initial data illustrating the extension of Le Jan-Sznitman techniques to stochastically explosive branching processes.

Theorem 3.2. *Suppose $u_0 \in [0,1)$. Then $\underline{u}(t) = \mathbb{E}(\underline{X}(t))$ is the unique solution to (2.2) in the class of solutions satisfying*

$$\|u(t)\|_\infty := \sup_{t\ge 0} |u(t)| < 1 \ .$$

Proof. The fact that when $u_0 \in [0,1]$, $\underline{u}(t) = \mathbb{E}(\underline{X}(t)) < \infty$ for all $t > 0$ follows from the definition of $\underline{X}$: (3.1). The uniqueness followes by the modified Le Jan-Sznitman argument. Namely, for a solution $u(t)$ consider the sequence of stochastic processes $Y_n(t)$ defined by (3.4). As noted in the proof of Theorem 3.1, by induction, $\mathbb{E}(X_n(t)) = u(t)$. Also, since $\|u\|_\infty < 1$, $Y_n(t) \to \underline{X}(t)$ a.s. as $n \to \infty$. Then, by the Lebesgue dominated convergence theorem, $u(t) = \underline{u}(t)$. Also recall that for $\alpha = 1$, the global solution exists if an only if the $0 \le u_0 \le 1$ □

Remark 3.1. As we have mentioned before, in the case $0 \le \alpha < 1$ it was shown in [6] (see β-field Burgers equation, $\beta = \sqrt{\alpha}$) that α-Riccati equation has a unique global in time solution for any initial data $u_0 \ge 0$.

Theorem 3.3. *Assume $\alpha \ge 5/2$ (i.e. $2\alpha - 1 \ge 4$). Then for any $u_0 \in [0, (2\alpha-1)/4)$ the minimal solution $\underline{u}(t) = \mathbb{E}(\underline{X}(t))$ is finite, and thus represents a global in time solution to (2.2) .*

(Recall that for any $\alpha \ge 0$ the solution $\underline{u}(t)$ is guaranteed to be finite for $u_0 \in [0,1]$.)

Proof. We estimate

$$\mathbb{E}(\underline{X}(t)) = \sum_{n=1}^{\infty} u_0^n P_n(t),$$

where $P_n(t)$ is defined by (2.11). Using the upper bound from (2.12), we get

$$\mathbb{E}(\underline{X}(t)) \le \sum_{n=1}^{\infty} u_0^n \frac{C_n}{(2\alpha-1)^{n-1}}\, e^{-t} \lesssim u_0 e^{-t} \sum_{n=1}^{\infty} \left(\frac{u_0}{2\alpha-1}\right)^n \frac{4^n}{\sqrt{\pi}\, n^{3/2}} \quad (\text{as } n\to\infty),$$

which converges when $u_0 \le (2\alpha-1)/4$, and thus the theorem follows. □

Remark 3.2. In fact, one can show that for all $n \ge 1$, the Catalan numbers satisfy $C_n \le \frac{4^n}{\sqrt{\pi} n^{3/2}}$. Therefore, if we denote

$$C_0 := \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \qquad C_0 < 3/2, \tag{3.5}$$

the proof of Theorem 3.3, yields the following estimate

$$\underline{u}(t) \le C_0 u_0 e^{-t}, \tag{3.6}$$

provided $u_0 \le \frac{2\alpha-1}{4}$.

Remark 3.3. It is interesting to explore the critical nature of $\alpha = 1$ by comparing the above existence result to the case $\alpha \in [0, 1]$ described in [7]. Generally speaking, one would expect that existence and uniqueness properties of (2.2) become better for smaller α. Indeed, for $\alpha < 1$, the Le Jan-Sznitman stochastic cascades are non-exploding and global solutions exist for arbitrary large initial data. In the case $\alpha = 1$ the equation becomes a basic logistic ODE; the cascade is still non-exploding, however, the global solutions exist if and only if $u_0 \in [0, 1]$. In the stochastically explosive regime $\alpha > 1$, like in the case $\alpha = 1$, we have global existence for small initial data (see Section 5 for the blow-up result), but the range of the initial data with global existence – see Theorems 3.3 and 3.2 – *grows* as $\alpha \to \infty$, even as underlying cascades become more explosive.

Remark 3.4. One can prove comparable results for existence of global solutions with small (but bigger then 1) initial data using fixed point methods. Such solutions will be unique in the class of solutions decaying fast enough to zero, and, by the minimality of $\underline{u}$, will coincide with $\underline{u} = \mathbb{E}(\underline{X})$. However, by Theorem 4.1 below, uniqueness fails if the time-decay assumption is removed, In particular, there exists a solution $\overline{u}(t)$ with $\liminf \overline{u} \geq 1$ as $t \to \infty$. It is not clear if there are other solutions with different behavior as $t \to \infty$.

4 An iterated system of stochastic processes – breakdown of global uniqueness.

Consider the following recursive system of stochastic processes associated with (2.2):

$$X_0(t)\text{– fixed}, \qquad X_n\left(t\right) = \begin{cases} u_0, & T_\emptyset \geq t, \\ X^{(1)}_{n-1}\left(\alpha(t - T_\emptyset)\right) X^{(2)}_{n-1}\left(\alpha(t - T_\emptyset)\right), & T_\emptyset < t \end{cases}, \; n \in \mathbb{N}, \tag{4.1}$$

where, as before, $X^{(1)}_{n-1}\left(\cdot\right)$ and $X^{(2)}_{n-1}\left(\cdot\right)$ are iid $X_{n-1}\left(\cdot\right)$.

Remark 4.1. By taking expected values, setting $y_j(t) := \mathbb{E}(X_n(t))$, we obtain

$$y_n(t) = u_0 e^{-t} + \int_0^t e^{-s} y^2_{n-1}(\alpha(t - s))\,\mathrm{d}s\,, \tag{4.2}$$

i.e. the above recursion corresponds to Picard-type iterations for (2.1)

Notice that in $X_0(t) = \delta$, then we can write explicitly:

$$X_n(t) = u_0^{N_n(t)} \delta^{M_n(t)},$$

where once again $N_n(t)$ represents number of branches that crossed time t containing *less* then n generations ($|v| < n$) and $M_n(t)$ is the number of all branches of n generations surviving by time t. Thus, in the case of no explosion by time t, i.e. if $S > t$, then after finitely many terms $X_n(t)$ will become equal to the $u_0^{N(t)}$, where $N(t)$ is the number of branches that cross t. Whenever $S < t$ (i.e. explosion by time t), due to Proposition 2.1 we have $M_n(t) \to \infty$, and therefore in this case:

- if $\delta > 1$, the sequence X_n converges to ∞.
- if $\delta = 1$, the sequence X_n is monotone. In fact, by Proposition 2.1, X_n is eventually constant (for fixed time t and outcome ω).
- if $\delta \in [0, 1)$, $X_n \to 0$ as $n \to \infty$.

Therefore the limit of X_n as $n \to \infty$ exists a.s, and collecting the above observations we obtain the following.

Proposition 4.1. *In the case $X_0(t) = \delta \in [0, 1)$ for all $t \geq 0$,*

$$\lim_{n\to\infty} X_n(t) = \underline{X}(t),$$

where $\underline{X}(t) < \infty$ a.s. – is the minimal solution stochastic process defined in (3.2).

If $X_0(t) = 1$ for all $t \geq 0$, then there exists $\overline{X}(t) < \infty$ a.s. such that

$$\lim_{n\to\infty} X_n(t) = \overline{X}(t).$$

Moreover, if $\overline{u}(t) = \mathbb{E}(\overline{X}(t)) \leq \Psi(t)$, where $\Psi : [0, \infty) \to [0, \infty)$ is a locally square integrable function, then $\overline{u}(t)$ is a solution of (2.2).

Proof. The convergence of $X_n(t)$ was addressed above. It remains to show that $\overline{u}(t)$ is a solution to (2.2). Indeed, in the case $X_0 = 1$, we have $X_n(t) \leq \max\{1, \overline{X}(t)\}$ a.s. for all $t \geq 0$. So $y_n(t) := \mathbb{E}(X_n(t)) \leq \max\{1, \overline{u}(t)\}$, and since $\overline{u}(t)$ is bounded by a locally square-integrable function, we can (by Lebesgue dominated convergence theorem) pass to the limit in (4.1) to conclude that $\overline{u}(t)$ is indeed a solution of (2.2). □

Note that solutions described in Remark 2.1 fit the framework of the Proposition above:

- in the case $u_0 = 1$, $\overline{X}(t) = 1$ (and thus $\overline{u}(t) = 1$) and $\underline{X} = \mathbb{1}_{S\geq t}$ with $\underline{u}(t) = \underline{u}_1(t) = \mathbb{P}(S \geq t)$.
- In the case $u_0 = 0$, $\underline{X}(t) = 0$ (thus $\underline{u}(t) = 0$) and $\overline{X} = \mathbb{1}_{L<t}$ with $\overline{u}(t) = \overline{u}_0(t) = \mathbb{P}(L \leq t)$.

Remark 4.2. Since for any $t > 0$ we have $\mathbb{P}(\{L < t\}) > 0$, for $X_0(t) = \delta > 1$:

$$\mathbb{E}\left(\lim_{n\to\infty} X_n(t)\right) = \infty.$$

Thus the case $\delta > 1$ is not relevant.

The next two theorems contain the main results of this section.

Theorem 4.1. *For any $u_0 \in [0, 1]$ there exist two different global solutions to (2.2):*

$$\underline{u}(t) = \mathbb{E}(\underline{X}(t)), \quad \textit{with the property } \lim_{t\to\infty} \underline{u}(t) = 0,$$

and

$$\overline{u}(t) = \mathbb{E}(\overline{X}(t)), \quad \textit{with the property } \lim_{t\to\infty} \overline{u}(t) = 1,$$

Proof. Since $0 \leq u_0 \leq 1$, we have that the expected values above are finite. We already noted in the proof of Theorem 3.2 that $\underline{u}$ is a solution to (2.2). Now, set $X_0(t) = 1$ for all $t \geq 0$. Then, by Proposition 4.1 $\lim_n X_n(t) = \overline{X}(t)$ a.s. and, since $X_n(t) \leq 1$, we have $\mathbb{E}(\overline{X}(t)) \leq 1$ – a locally square-integrable upper bound. Thus by Proposition 4.1 $\overline{u}(t)$ is also a solution to (2.2). In fact, due to the explosion,

$$0 \leq \underline{u}(t) \leq \mathbb{P}(S \geq t) \to 0, \quad \text{as } t \to \infty,$$

and

$$1 \geq \overline{u}(t) \geq \mathbb{P}(L < 1) \to 1, \quad \text{as } t \to \infty,$$

which finishes the proof. □

Note: The above theorem does not contradict the uniqueness result from Theorem 3.2 since $\|\overline{u}\|_\infty = 1$.

Recall that when $\alpha > 5/2$,Theorem 3.3 establishes existence of global solutions for a range of initial data $u_0 > 1$, specifically for the initial data $1 < u_0 \le (2\alpha - 1)/4$. Next, we will show that uniqueness fails in the case $u_0 > 1$ as well with comparable upper bound on u_0. As in Theorem 4.1, the second solution will turn out to be $\overline{u}(t)$, however the the proof of $\overline{u}(t) < \infty$ is more involved in the case $u_0 > 1$.

Theorem 4.2. *Let $\alpha > 5/2$. Denote*

$$\phi_\infty(\alpha) := \frac{6\alpha^2 - 15\alpha + 4}{4(\alpha-1)(2\alpha-1)}, \qquad \left(\frac{1}{6} \le \phi_\infty(\alpha) < \frac{3}{4}\right).$$

Assume

$$1 < u_0 \le \frac{2\alpha - 1}{4} - \phi_\infty(\alpha).$$

Then the system (2.2), in addition to the lower solution $\underline{u}(t)$, admits the solution $\overline{u}(t) = \mathbb{E}(\overline{X}(t))$ for which the following estimate holds

$$\underline{u}(t) < \overline{u}(t) \le \underline{w}(t) + (1 - e^{-t}) \qquad \textit{for all } t > 0, \tag{4.3}$$

where $\underline{w}(t)$ is a lower solution to (2.2) with initial data w_0 satisfying

$$u_0 + \phi_\infty(\alpha) \le w_0 \le \frac{2\alpha - 1}{4}.$$

Proof. Recall that $\overline{X}(t)$ can be obtained as the limit of the Le Jan-Sznitman iteration (4.1) with initialization $X_0(t) = 1$ for $t \ge 0$. Note that in this case for a fixed t and ω, $X_n(t)$ is increasing, the (possibly infinite) limit $\overline{X}(t) = \lim_{n\to\infty} X_n(t)$ exists and, by the monotone convergence theorem:

$$\overline{u}(t) = \mathbb{E}(\overline{X}(t)) = \lim_{n\to\infty} \mathbb{E}(X_n(t)) \in [0, \infty] \qquad \text{for al } t \ge 0.$$

Note that in that $\underline{X}(t) \le \overline{X}(t)$ a.s, and in the case $\omega \in \{L < t\}$ we have

$$X_n(t)(\omega) = 1, \text{and thus } \overline{X}(t)(\omega) = 1,$$

while $\underline{X}(\omega) = 0$ (see (3.2)). Therefore, since for any $t > 0$ and $\epsilon \in (0, t)$, $\mathbb{P}(L < t - \epsilon) > 0$, we conclude

$$\overline{u}(t) > \underline{u}(t) \qquad \text{for all } t > 0.$$

Next, we will show that for suitable $\underline{w}$ the upper bound

$$\mathbb{E}(X_n(t)) \le \underline{w}(t) + (1 - e^{-t}) \tag{4.4}$$

holds for for any $n \ge 1$. This will allow us to use the dominated convergence theorem to pass to the limit as $n \to \infty$ and conclude by Proposition 4.1 that $\overline{u}(t)$ is in fact solution to (2.2) satisfying conditions of the Theorem.

To prove (4.4) proceed by induction.

In case $n = 1$

$$\mathbb{E}(X_1(t)) = u_0 e^{-t} + e^{-t} \int_0^t e^s\, \mathbb{E}^2(X_0(\alpha s))\, \mathrm{d}s,$$

and since $\underline{w}(t) \geq w_0 e^{-t} > u_0 e^{-t}$ (see e.g. Theorem 5.1 below), and $\mathbb{E}(X_0) = 1$, we have that (4.4) holds for $n = 1$.

Assume (4.4) holds for some n. Then

$$\mathbb{E}(X_{n+1}(t)) = u_0 e^{-t} + e^{-t} \int_0^t e^s\, \mathbb{E}^2(X_n(\alpha s))\, \mathrm{d}s \leq u_0 e^{-t} + e^{-t} \int_0^t e^s \,\left(\underline{w}(\alpha s) + (1 - e^{-\alpha s})\right)^2 \,\mathrm{d}s$$

$$\leq u_0 e^{-t} + e^{-t} \int_0^t e^s\, \underline{w}^2(\alpha s)\, \mathrm{d}s + 2e^{-t} \int_0^t e^s\, \underline{w}(\alpha s)\,(1 - e^{-\alpha s})\, \mathrm{d}s + e^{-t} \int_0^t e^s \,\left(1 - 2e^{-\alpha s} + e^{-2\alpha s}\right)\, \mathrm{d}s.$$

Since

$$e^{-t} \int_0^t e^s\, \underline{w}^2(\alpha s)\, \mathrm{d}s = \underline{w}(t) - w_0 e^{-t},$$

$$\underline{w}(t) \leq C_0 w_0 e^{-t} \qquad \text{(see (3.6))},$$

and

$$e^{-t} \int_0^t e^s \,\left(1 - 2e^{-\alpha s} + e^{-2\alpha s}\right)\, \mathrm{d}s = 1 - e^{-t} - e^{-t} \int_0^t \left(2e^{-(\alpha-1)s} - e^{-(2\alpha-1)s}\right) \mathrm{d}s,$$

we conclude that

$$\begin{aligned}
\mathbb{E}(X_{n+1}(t)) &\leq \underline{w}(t) + (1 - e^{-t}) - (w_0 - u_0)e^{-t} \\
&\quad + 2C_0 w_0\, e^{-t} \int_0^t \left(e^{-(\alpha-1)s} - e^{-(2\alpha-1)s}\right) \mathrm{d}s - e^{-t} \int_0^t \left(2e^{-(\alpha-1)s} - e^{-(2\alpha-1)s}\right) \mathrm{d}s \\
&= \underline{w}(t) + (1 - e^{-t}) - e^{-t}\, \Gamma(t),
\end{aligned}$$

where

$$\Gamma(t) = w_0 - u_0 - (2C_0 \underline{w}_0 - 2) \int_0^t e^{-(\alpha-1)s}\, \mathrm{d}s + (2C_0 \underline{w}_0 - 1) \int_0^t e^{-(2\alpha-1)s}\, \mathrm{d}s.$$

To show that w_0 can be chosen such that $\Gamma(t) \geq 0$, consider its derivative

$$\Gamma'(t) = -ae^{-(\alpha-1)t} + be^{-(2\alpha-1)t}, \quad \text{where } a = 2C_0 w_0 - 2,\ b = 2C_0 w_0 - 1.$$

We have $\Gamma'(0) = b - a = 1 > 0$, $\lim_{t\to\infty} \Gamma'(t) = 0$. Moreover there exists unique $t_0 > 0$ such that $\Gamma'(t_0) = 0$:

$$-ae^{-(\alpha-1)t_0} = be^{-(2\alpha-1)t_0}, \qquad \text{i.e,} \quad e^{-t_0} = \left(\frac{a}{b}\right)^{1/\alpha}$$

This means that $\Gamma(t)$ attains a global maximum on $[0, \infty)$ at $t = t_0$ and

$$\inf_{[0,\infty)} \Gamma(t) = \min\{\Gamma(0), \Gamma_\infty\},$$

where $\Gamma(0) = w_0 - u_0$ and

$$\Gamma_\infty = \lim_{t\to\infty} \Gamma(t) = w_0 - u_0 - \frac{a}{\alpha - 1} + \frac{b}{2\alpha - 1}.$$

Since $b = a + 1$ we write

$$-\frac{a}{\alpha-1} + \frac{b}{2\alpha-1} = \frac{1}{2\alpha-1} - \frac{a\alpha}{(\alpha-1)(2\alpha-1)} = \frac{1}{2\alpha-1}\left(1 - \frac{2\alpha(C_0 w_0 - 1)}{\alpha-1}\right)$$
$$\geq \frac{1}{2\alpha-1}\left(1 - \frac{2\alpha\left(\frac{3}{2}\frac{2\alpha-1}{4} - 1\right)}{\alpha-1}\right) = \frac{-6\alpha^2 + 15\alpha - 4}{4(\alpha-1)(2\alpha-1)} = -\phi_\infty.$$

In the above we used the estimate $C_0 \leq 3/2$ (see (3.5)) and $w_0 \leq (2\alpha-1)/4$.

Elementary analysis shows that $\phi_\infty(\alpha)$ is increasing for $\alpha \geq 5/2$, and $1/6 \leq \phi_\infty(\alpha) < 3/4$ for all $\alpha \geq 5/2$. Therefore $\Gamma(t) \geq 0$ for all $t \geq 0$ provided w_0 is chosen such that

$$u_0 + \phi_\infty \leq w_0 \qquad \left(\text{Recall that we also needed } w_0 \leq \frac{2\alpha-1}{4}\right).$$

Since $\Gamma(t) \geq 0$ implies (4.4) holds for $n+1$, we obtain that it holds for any n by induction, and thus the Theorem follows.

□

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5 Lack of global existence for large initial data.

In this section we present a probabilistic approach, using Le Jan-Sznitman cascade, to show lack of existence of global in time solutions and compare it to a traditional approach based on the direct (elementary) analysis of the differential equation (2.2).

5.1 Blow-up, stochastic cascade approach.

Consider the Le Jan-Sznitman tree introduced in Section 2. Recall that by Lemma 2.1, the chance of non-explosion by time $t > 0$ is positive, and therefore, it make sense to consider finiteness of the minimal solutions $\mathbb{E}(\underline{X}(t))$ for initial data of size. The main result of this section is that in fact big initial data necessarily lead to finite-time blow up.

Theorem 5.1. *For any $u_0 \geq 0$.*

$$\underline{u}(t) = \mathbb{E}(\underline{X}(t)) \geq u_0 e^{-t} \frac{(2\alpha-1)e^{(2\alpha-1)t}}{(2\alpha-1-u_0)e^{(2\alpha-1)t} + u_0}. \tag{5.1}$$

As a consequence, if $u_0 \geq 2\alpha - 1$, then any solution of (2.2) blows-up (cease to exist) in finite time, with blow-up time

$$T_{\text{bu}} < \frac{1}{2\alpha-1}\ln\left(\frac{u_0}{u_0 - 2\alpha + 1}\right).$$

Proof. Using (2.12) we obtain:

$$\mathbb{E}(\underline{X}(t)) = \sum_{n=1}^{\infty} u_0^n P_n(t) \geq \sum_{n=1}^{\infty} \frac{u_0^n}{(2\alpha-1)^{n-1}} e^{-t}\left(1 - e^{-(2\alpha-1)t}\right)^{n-1} = u_0 e^{-t}\sum_{k=0}^{\infty}\left(\frac{u_0\left(1 - e^{-(2\alpha-1)t}\right)}{2\alpha-1}\right)^k.$$

Thus, as long as $t > 0$ is small enough, we can add the power series to obtain

$$\mathbb{E}(\underline{X}(t)) \geq u_0 e^{-t}\frac{1}{1 - \left(\frac{u_0\left(1-e^{-(2\alpha-1)t}\right)}{2\alpha-1}\right)} = u_0 e^{-t}\frac{(2\alpha-1)e^{(2\alpha-1)t}}{(2\alpha-1-u_0)e^{(2\alpha-1)t} + u_0},$$

and so (5.1) holds. Clearly, if $u_0 \geq 2\alpha - 1$ the lower bound becomes infinite at $\frac{1}{2\alpha-1} \ln\left(\frac{u_0}{u_0-2\alpha+1}\right)$, and therefore the solution cannot exist past that time.

□

Remark 5.1. In fact, from (2.1) we have that the derivative would cease to exist beyond $t = T_{\text{bu}}/\alpha$ and

$$\int_0^{T_{\text{bu}}} u^2(s)\,\mathrm{d}s < \infty.$$

5.2 Blow-up – a direct approach.

Let

$$w(t) = u(t)e^t \ .$$

Then w satisfies

$$\left\| \begin{array}{l} w'(t) = w^2(\alpha t)\, e^{-(2\alpha-1)t} \\ w(0) = u_0(\geq 0) \end{array} \right. \qquad . \tag{5.2}$$

Note that w is non-negative and increasing for $t \geq 0$.

We have can prove the following bound for w.

Proposition 5.1. *For any $u_0 \geq 0$.*

$$w(t) \geq u_0 \frac{(2\alpha-1)e^{(2\alpha-1)t}}{(2\alpha-1-u_0)e^{(2\alpha-1)t} + u_0} \ . \tag{5.3}$$

As a consequence, if $u_0 > 2\alpha - 1$, then any solution w of (5.2) blows-up in finite time, with blow-up time

$$\tilde{T}_{\text{bu}} < \frac{1}{2\alpha-1} \ln\left(\frac{u_0}{u_0 - 2\alpha + 1}\right).$$

Proof. Note that for form (5.2) $w'(t) \geq 0$, so $w(t)$ is increasing. Since $\alpha > 1$, over the interval of existence of w, we have:

$$w'(t) \geq w^2(t)\, e^{-(2\alpha-1)t},$$

and so

$$\int_0^t \frac{w'(s)}{w^2(s)}\,\mathrm{d}s = \int_0^t e^{-(2\alpha-1)s}\,\mathrm{d}s,$$

which implies (5.3).

□

Observe that since $w(t) = u(t)e^t$, the lower bound above leads to exactly the same estimate as in (5.1), which was obtained using the stochastic cascade approach.

We conclude this section by summarizing the existence and uniqueness theory of global in time solutions of the α-Riccati equation (2.1) from Theorems 3.2, 3.3, 4.1, 4.2, 5.1 and Remark 3.1 in the following figure (see fig. 2).

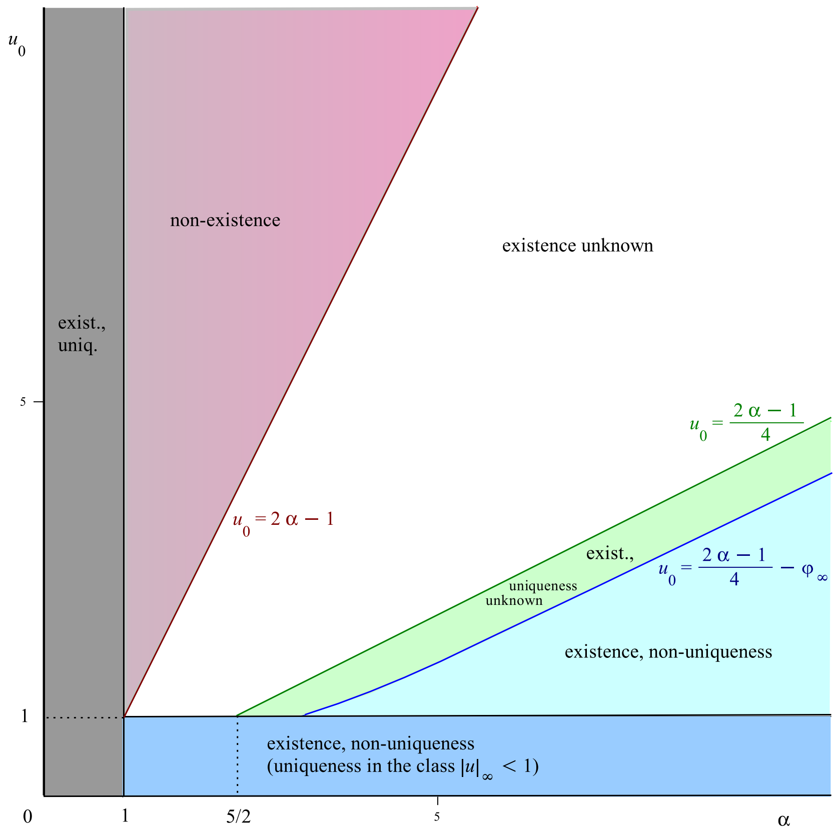

Figure 2: Existence and uniqueness properties of global in time solutions to (2.2).

6 Long-time behavior of global solutions.

We start by the following application of the Le Jan-Sznitman iterations to study long-time behavior of the global solutions.

Proposition 6.1. *Let $\alpha > 1$ and assume $u(t)$ is a global in time solution to (2.1). Then*

$$\liminf_{t\to\infty} u(t) \le 1. \tag{6.1}$$

Proof. Recall, from the proof on Theorem 3.1, with initialization $Y_0(t) = u(t)$, the Le Jan-Sznitman iterations (3.4) satisfy $\mathbb{E}(Y_n(t)) = u(t)$ for all n and t. Also note that for all $t > 0$ and any $\epsilon \in (0, t)$, $\mathbb{P}(L < t - \epsilon) > 0$. If $\omega \in \{L < t - \epsilon\}$ we can compute explicitly

$$Y_n(t)(\omega) = \prod_{|v|=n} u(\alpha^n(t - \theta_v(\omega))). \tag{6.2}$$

Now, assume by contradiction, $\liminf_{t\to\infty} u(t) > 1$. Then there exists $K_0 > 1$ and $t_0 > 0$ such that $u(t) \ge K_0$ for all $t \ge t_0$.

Fix $t > 0$. If $\omega \in \{L < t - \epsilon\}$, then there exists an $n_0 > 0$ such that for all $n \ge n_0$ we have that if v is a branch with $|v| = n$, then $\alpha^n(t - \theta_v) \ge \alpha^n(t - L) = \alpha^n\epsilon > t_0$, and therefore (6.2) yields $Y_n(t)(\omega) \ge K_0^{2^n} \to \infty$ as $n \to \infty$. This means that for $n \ge n_0$

$$u(t) = \mathbb{E}(Y_n(t)) \ge K_0^{2^n}\mathbb{P}(L < t - \epsilon) \to \infty \qquad \text{as } n \to \infty,$$

i.e. $u(t) = \infty$, which contradict the assumption that u is a global solution. Consequently, (6.1) must hold. □

We can also show boundedness of the upper limit, this time using analytical techniques. We start by observing that $v(t) = e^t u(t)$ satisfies the equation

$$v'(t) = e^{-(2\alpha-1)t} v(\alpha t), \qquad v(0) = u_0.$$

Thus, $v(t)$ is increasing, and so for any $t, s > 0$

$$\begin{aligned} u(t) &\geq u(s)\, e^{s-t}, \qquad \text{if } t > s, \\ u(t) &\leq u(s)\, e^{s-t}, \qquad \text{if } t < s. \end{aligned} \tag{6.3}$$

Proposition 6.2. *Let $\alpha > 1$ and $u(t)$ – a solution to (2.1) defined for all $t \geq 0$. Then*

$$\limsup_{t\to\infty} u(t) < \infty. \tag{6.4}$$

Proof. By contradiction, assume

$$\limsup_{t\to\infty} u(t) = \infty\,. \tag{6.5}$$

Then, there exists a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} u(t_n) = \infty \qquad \text{and} \qquad u(t) \leq u(t_n) \quad \text{whenever } t \leq t_n. \tag{6.6}$$

Now, fix a $\beta > 1$. By (6.3) we have

$$u(t) \geq u(t_n)\, e^{-(t-t_n)} \qquad \text{for all } t \geq t_n\,.$$

Therefore, if $0 \leq t - t_n \leq \ln\beta$, then

$$u(t) \geq \frac{M_n}{\beta}, \qquad M_n := u(t_n)\,.$$

Note that by (6.5) $M_n \to \infty$, as $n \to \infty$. Then, looking back at t_n/α, we have for $t \in [t_n/\alpha, t_n/\alpha + \ln\beta/\alpha]$,

$$u'(t) = -u(t) + u^2(\alpha t) \geq -M_n + \left(\frac{M_n}{\beta}\right)^2,$$

and therefore

$$u(t_n/\alpha + \ln\beta/\alpha) \geq \left(-M_n + \left(\frac{M_n}{\beta}\right)^2\right)\frac{\ln\beta}{\alpha} =: \Lambda_n.$$

Also, since $t_n \to \infty$, $t_n > t_n/\alpha + \ln\beta/\alpha$ for n big enough. Since $M_n \to \infty$, $\Lambda_n > M_n$ for n big enough. This means that for n big enough $u(t_n/\alpha + \ln\beta/\alpha) > M_n = u(t_n)$, which contradicts the choice of t_n to satisfy the second part of (6.6).

Consequently, M_n is not convergent for infinity, which means (6.4) holds.

□

Remark 6.1. In the case when there exists

$$L = \lim_{t\to\infty} u(t),$$

the results in this section show $L \leq 1$, and using (2.1) one can easily see that

$$L \in \{0, 1\} \; - \; \text{one of the two stationary solutions for (2.1).}$$

7 Local in time existence and lack of uniqueness.

As noted in previous section, global in time solutions do not exist for sufficiently large initial data ($u_0 > 2\alpha - 1$) in the case $\alpha > 1$. (As we have mentioned before, for $\alpha < 1$ all the solutions are global.) The question of existence of local in time solutions when $\alpha > 0$ is not straightforward due to the non-local character of (2.1): the behavior of a solution at time t depends on the "future" time αt.

By *local in time solution on* $[0, T)$ of (2.1) (or, equivalently of (2.2)) we understand a function $u(t)$ defined on the interval $[0, \alpha T)$ that satisfies (2.1) (or (2.2)) for all $t \in [0, T)$. Note that $T = \infty$ corresponds to a global in time solution.

We can attempt to construct a local solution via the same iterative procedures described in Section 4 as well as in the proof of Theorem 3.1. The key observation is that, e.g., in the case of $X_0(t) = 0$ for $t \geq 0$, if $\underline{u}(t) = \lim_{n\to\infty} \mathbb{E}(X_n(t)) < \infty$ for $t \in [0, T)$, then $\underline{u}(t)$ is a local solution to (2.2) on $[0, T/\alpha)$. Just like in the global in time case, by choosing various initializations $X_0(t)$, we can potentially create other local solutions for the same initial data, provided we can pass to the limit as $n \to \infty$ in (4.2). One key difference is that the local existence time T may a priori be different for different (local) solutions corresponding to the same initial data u_0.

We will be interested to estimate *the maximal existence time*:

$$T(\alpha, u_0) = \sup\{T \geq 0 : \text{ there exists a local in time solution } u(t) \text{ on } [0, T)\}. \tag{7.1}$$

Note that this maximal local existence time $T(\alpha, u_0)$ cannot be obtained by attempting to extend an arbitrary local solution to a maximal existence time due to possible lack of uniqueness (which in fact will be established later in this section). Nevertheless, the following version of Theorem 3.1 holds in the locally in time.

Theorem 7.1. *For any $u_0 \geq 0$ and $\alpha \geq 0$, if $u(t)$ is a (local) solution of (2.2) on $[0, T)$, then*

- $\mathbb{E}(\underline{X}(t)) < \infty$ *for* $t \in [0, T)$,
- $\underline{u}(t) = \mathbb{E}(\underline{X}(t))$ *is a local solution to (2.2) on (at least)* $[0, T)$,
- $u(t) \geq \underline{u}(t)$ *for all* $t \in [0, T)$.

Proof. The proof is a straightforward adaptation of the proof of Theorem 3.1. □

Remark 7.1. Theorem 7.1 allows us once more to call $\underline{u}(t)$ *the minimal local solution*, and, as a consequence, the maximal existence time of $\underline{u}(t)$, i.e. the maximal interval $[0, T^*)$ were $\mathbb{E}(\underline{X}(t)) < \infty$, is precisely $T^*/\alpha = T(\alpha, u_0)$ – the maximal existence time from (7.1). In particular, this shows that if $T(\alpha, u_0) = \infty$, then there exists a *global solution* for this u_0 and α – namely $u(t) = \underline{u}(t)$.

Observe that the lower bounds on $P_n(t)$ from Lemma 2.1 allows the following estimate on $T(\alpha, u_0)$:

Lemma 7.1. *Let $\alpha > 1$ and $u_0 \geq 2\alpha - 1$. Then*

$$T(\alpha, u_0) \leq \frac{1}{2\alpha - 1} \ln\left(\frac{u_0}{u_0 - (2\alpha - 1)}\right) \tag{7.2}$$

Proof. According to Remark 7.1, we need to estimate the interval where $\underline{u}(t) < \infty$. But, analogously to the proof on Theorem 5.1, using the lower bound from (2.12), we get

$$\mathbb{E}(\underline{X}(t)) = \sum_{n=1}^{\infty} u_0^n P_n(t) \geq \sum_{n=1}^{\infty} \frac{u_0^n}{(2\alpha - 1)^{n-1}}\, e^{-t} \left(1 - e^{-(2\alpha-1)t}\right)^{n-1} = \frac{u_0 e^{-t}}{1 - \frac{u_0}{2\alpha-1}(1 - e^{-(2\alpha-1)})}.$$

Solving for t, we obtain $\underline{u}(t) < \infty$ implies (7.2). □

To obtain a lower bound for $T(\alpha, u_0)$ we will introduce a series of iterative estimates based on the following *backwards propagation procedure*.

Backwards propagation.

Suppose $u(t)$ is a solution to (2.1) and let

$$v(t) = u(t)e^t.$$

Then, as we have noted in Section 6, v satisfies

$$\left\| \begin{array}{l} v'(t) = e^{-(2\alpha-1)t} v^2(\alpha t) \\ v(0) = u(0) = u_0 \end{array} \right. . \tag{7.3}$$

Choose $j_0 \in \mathbb{Z}$ and let $f : [\alpha^{j_0}, \alpha^{j_0+1}] \to [0, \infty)$ be an arbitrary continuous function. Set

$$v_{[j_0]}(t) := f(t).$$

For $j < j_0$ define inductively

$$v_{[j-1]}(t) := v_{[j]}(\alpha^j) - \int_t^{\alpha^j} e^{-(2\alpha-1)s} v_{[j]}^2(\alpha s)\, \mathrm{d}s. \tag{7.4}$$

Finally, define a function $v_f : (0, \alpha^{j_0+1}) \to \mathbb{R}$ by

$$v_f(t) := v_{[j]}(t) \qquad \text{if } t \in [\alpha^j, \alpha^{j+1}],\ j \le j_0. \tag{7.5}$$

Proposition 7.1. *For any $j_0 \in \mathbb{Z}$ and any $f : [\alpha^{j_0}, \alpha^{j_0+1}] \to [0, \infty)$, the function v_f defined by (7.5) is a local solution on $[0, \alpha^{j_0+1})$ to (7.3) with initial data*

$$u_0 := \lim_{t \to 0} v_f(t) \in [-\infty, f(\alpha^{j_0})\,).$$

Proof. First observe that v_f is differentiable on $(0, \alpha^{j_0})$, in particular, using (7.4) we notice for any $j \le j_0$ $v_f'(\alpha_+^{j-1}) = v_{[j-1]}'(\alpha_+^{j-1}) = e^{-(2\alpha-1)\alpha^{j-1}} v_{[j]}^2(\alpha^j) = e^{-(2\alpha-1)\alpha^{j-1}} v_{[j-1]}^2(\alpha^j) = v_{[j-2]}'(\alpha_-^{j-1}) = v_f'(\alpha_-^{j-1})$, which also shows that the differential equation is satisfied at $t = \alpha^{j-1}$. Clearly, the equation is also satisfied for $t \neq \alpha^{j-1}$, $0 < t < \alpha^{j_0}$.

Second, the differential equation in (7.3) implies $v_f(t)$ is decreasing for $t < \alpha^{j_0}$, so the limit as $t \to 0$ exists in $[-\infty, f(\alpha^{j_0})\,)$.

$\square$

Remark 7.2. Clearly, the backwards propagation procedure described above provides a recipe for construction of multiple local solutions by choosing different functions $f(x)$ and $j_0 \in \mathbb{Z}$. The challenge is to find suitable choices for f and j_0 such that the $\lim_{t \to 0} v_f(t)$ satisfies a given initial condition $u_0 \ge 0$.

Remark 7.3. One may attempt to propagate $f(x)$ *forwards* for $j > j_0$ by setting

$$v_{[j+1]}(t) = \left(e^{-(2\alpha-1)t/\alpha} v_{[j]}'(t/\alpha) \right)^{1/2}, \qquad t \in [\alpha^{j+1}, \alpha^{j+2}],$$

but compatibility conditions

$$v_{[j]}'(\alpha^j) = e^{(2\alpha-1)\alpha^j} v_{[j]}^2(\alpha^{j+1})$$

may not hold for all $j > j_0$. Thus such a forward propagation might not lead to a global solution.

Discrete iterated estimate scheme using backward propagation.

In order to estimate $T(\alpha, u_0)$ from below, we need to obtain a bound on $u_0 := \lim_{t\to 0} v_f(t)$ in terms of f and j_0.

Denote

$$M_{j_0} := \sup\{f(x) : x \in [\alpha^{j_0}, \alpha^{j_0+1}]\}, \qquad m_{j_0} := \inf\{f(x) : x \in [\alpha^{j_0}, \alpha^{j_0+1}]\}$$

Next, for all $j \le j_0$, set

$$m_{j-1} := m_j - M_{j_0}^2 \int_{\alpha^{j-1}}^{\alpha^j} e^{-(2\alpha-1)s}\, \mathrm{d}s. \tag{7.6}$$

Note that since $v_f(t)$ is decreasing for $t < \alpha^{j_0}$, by induction, $v_f(\alpha^j) \ge m_j$ for all $j \le j_0$. Therefore,

$$u_0 := \lim_{t\to 0} v_f(t) \ge \lim_{j\to -\infty} m_j =: l_0,$$

where the last limit exists since m_j decreasing with $-j$. In fact, clearly,

$$l_0 = m_{j_0} - M_{j_0}^2 \int_0^{\alpha^{j_0}} e^{-(2\alpha-1)s}\, \mathrm{d}s = m_{j_0} - M_{j_0}^2\, \frac{1 - e^{-(2\alpha-1)\alpha^{j_0}}}{2\alpha - 1}.$$

Thus we have established the following result.

Proposition 7.2. *Let* $u_0 = \lim_{t\to 0} v_f(t)$. *Then*

$$u_0 \ge m_{j_0} - M_{j_0}^2\, \frac{1 - e^{-(2\alpha-1)T_0}}{2\alpha - 1}.$$

where

$$T_0 := \alpha^{j_0}.$$

Note that the biggest lower bound on L_0 above is achieved when

$$m_{j_0} = M_{j_0} = \frac{1}{2}\, \frac{2\alpha - 1}{1 - e^{-(2\alpha-1)T_0}},$$

i.e. for the constant function $f(t) = M_{j_0}$, with M_{j_0} given above. This choice of f generates a local solution v_f corresponding to the initial data

$$u_0 \ge \frac{2\alpha - 1}{4} \frac{1}{1 - e^{-(2\alpha-1)T_0}}. \tag{7.7}$$

Solving for T_0 we get

$$T_0 \ge \frac{1}{2\alpha - 1}\, \ln\left(\frac{u_0}{u_0 - \frac{2\alpha-1}{4}}\right),$$

whenever $u_0 > \frac{2\alpha-1}{4}$.

Thus, recalling that $u(t) = v_f(t)e^{-t}$, we obtain the following Lemma.

Lemma 7.2. *Let* $\alpha > 1$ *and* $u_0 \ge \frac{2\alpha-1}{4}$. *Then*

$$T(\alpha, u_0) \ge \frac{1}{2\alpha - 1}\, \ln\left(\frac{u_0}{u_0 - \frac{2\alpha-1}{4}}\right). \tag{7.8}$$

Non-uniqueness of local solutions.

The idea of backward propagation and the discrete iterated scheme described above provides a simple way to prove non-uniqueness of local solutions.

We start with the lemma.

Lemma 7.3. *Let v_1 and v_2 be two local solutions of (7.3) such that for some $t_0 > 0$, $v_1(t_0) = v_2(t_0)$. Then $v_1(0) = v_2(0)$ (i.e. both solutions correspond to the same initial data.)*

Proof. Note that if $v_1(t) > v_2(t)$ for all $t \in [\alpha^j, \alpha^{j+1}]$ and some $j \in \mathbb{Z}$, then $v_1(t) > v_2(t)$ for all $t \geq \alpha^j$ within their common interval of validity.

Thus, two solutions that intersect at some $t_0 > 0$, must intersect on any interval $[\alpha^j, \alpha^{j+1}]$ such that $\alpha^j < t_0$. Since both solutions are decreasing, these intersections imply the solutions correspond to the same initial condition. □

Theorem 7.2. *Local solutions for (2.1) are not unique.*

Proof. We now can build local solutions that come form the same initial data using backwards propagation by starting with $f_{(1)}$ and $f_{(2)}$ on $[\alpha^{j_0}, \alpha^{j_0+1}]$ such that

$$m^{(1)}_{j_0}, m^{(2)}_{j_0}, M^{(1)}_{j_0}, M^{(2)}_{j_0} \lesssim \frac{1}{2} \frac{2\alpha - 1}{1 - e^{-(2\alpha-1)\alpha^{j_0}}} =: \bar{M}_{j_0},$$

such that $v_{f_{(1)}}(t)$ and $v_{f_{(2)}}(t)$ intersect at a $t_0 \in [\alpha^{j_0-1}, \alpha^{j_0})$. Clearly uncountable many choices of $f_{(1)}$ and $f_{(2)}$ defined on $[\alpha^{j_0}, \alpha^{j_0+1}]$ will force an intersection of the solution on the interval $[\alpha^{j_0-1}, \alpha^{j_0}]$ using the recurrence (7.4). The upper bound above together with (7.7) ensures $v_{f_{(1)}}(0)$ and $v_{f_{(2)}}(0)$ positive. By Lemma 7.3, $v_{f_{(1)}}(0) = v_{f_{(2)}}(0)$. □

8 Duality between $\underline{u}$ for $u_0 = 1$ and $\overline{u}$ for $u_0 = 0$.

Note that in the case $u_0 = 0$, we can view (2.2) as a convolution equation

$$u(t) = \left[u^2(\alpha\,\cdot) * e(\cdot)\right](t), \quad t \geq 0, \tag{8.1}$$

where $e(s) = e^{-s}, s \geq 0$. On the other hand, for $u_0 = 1$, letting $v = 1 - u$, one also obtains a (dual) convolution equation of the form

$$v(t) = \left[(2v(\alpha\,\cdot) - v^2(\alpha\,\cdot)) * e(\cdot)\right](t), \quad t \geq 0. \tag{8.2}$$

Following [2], note that if $U^{(1)}, U^{(2)}$ are i.i.d. random variables distributed as a non-negative random variable U, and independent of a mean one exponentially distributed random variable T_0, then for $u_M(t) = \mathbb{P}(U \leq t), t \geq 0$, (8.1) is the (weak) distributional form of the stochastic equation

$$U \overset{\text{dist}}{=} T_0 + \frac{1}{\alpha} U^{(1)} \vee U^{(2)}, \tag{8.3}$$

where $a \vee b = \max\{a, b\}$. Similarly, if $V^{(1)}, V^{(2)}$ are i.i.d. random variables distributed as a non-negative random variable V, and independent of a mean one exponentially distributed random variable T_0, then for $v_m(t) = \mathbb{P}(V \leq t), t \geq 0$, the inclusion-exclusion formula shows that (8.2) is the (weak) distributional form of the stochastic equation

$$V \overset{\text{dist}}{=} T_0 + \frac{1}{\alpha} V^{(1)} \wedge V^{(2)}, \tag{8.4}$$

where $a \wedge b = \min\{a,b\}$. Correspondingly, $u_m(t) = 1 - v_m = \mathbb{P}(V > t)$ solves (2.2) with the initial condition $u_0 = 1$.

Observe that L and S solve (8.3), (8.4), respectively. In particular

$$u_M(t) = \mathbb{P}(L \le t), \quad u_m(t) = 1 - \mathbb{P}(S \le t) = \mathbb{P}(S > t), \quad t \ge 0,$$

solve (8.1) and (8.2) respectively. Note that in the terms of the notations in previous sections, $u_M(t) = \overline{u}(t)$ corresponding to $u_0 = 0$ and $u_m(t) = \underline{u}(t)$ corresponding to $u_0 = 1$.

The duality of these solutions is captured in the following theorem.

Theorem 8.1. *There is a dual, in the sense of relations (8.3), (8.4), family of solutions u_M, u_m to (2.2).*

1. *If $0 \le \alpha \le 1$ the constant solution $u(t) = u_0, t \ge 0$, is unique for either choice of $u_0 \in \{0,1\}$. In particular, $S = L = \infty$ a.s, and $u_M(t) = \mathbb{P}(L \le t) = 0, t \ge 0$, and $u_n(t) = \mathbb{P}(S > t) = 1, t \ge 0$.*

2. *If $\alpha > 1$ we L and S are nontrivial solutions to the dual equations (8.3), (8.4) giving raise to the non-constant solutions $u_M(t) = \mathbb{P}(L \le t), t \ge 0$, and $u_m(t) = 1 - \mathbb{P}(S \le t) = \mathbb{P}(S > t), t \ge 0$, are non-constant solutions for the cases $u_0 = 0, 1$, respectively.*

3. *In the case $\alpha > 1$, u_M and u_m are, respectively, maximal and minimal in the space of non-negative solutions on $[0, \infty)$ bounded by one.*

Proof. The part 1 follows from the non-explosion of the Le Jan-Sznitman stochastic process (Theorem 2.1) The uniqueness was proven in [7] but it also follows from part 3.

Part 2 follows again Theorem 2.1, using the above-mentioned observation that $\mathbb{P}(L \le t)$ and $\mathbb{P}(S \le t)$ solve (in distribution) (8.3), (8.4) respectively.

The maximality of u_M was also established in [2] using the monotonicity (increasing) of the operator $u \to Au\,(t) = \int_0^t u^2(\alpha(t-s))e^{-s}\mathrm{d}s, t \ge 0$. In particular for any solution u with values in $[0,1]$, one has $u = Au \le \int_0^t e^{-s}ds = 1 - e^{-t}, t \ge 0$. Iterating yields $u \le A^n F, n = 1, 2, \ldots$, for $F(t) = 1 - e^{-t}, t \ge 0$. Thus

$$u \le \lim_{n\to\infty} A^n F.$$

Note that $L_n = \max_{|v|=n} \sum_{j=0}^n \alpha^{-j} T_{v|j}$ can be obtained (in distribution) by iterating $U_0 = 0$ $n+1$ times via (8.3), and thus, its distribution function is $A^{n+1}1\,(t) = A^n F\,(t)$. Thus, since, $L = \lim_{n\to\infty} L_n$, $\lim_{n\to\infty} A^n F\,(t) = \mathbb{P}(L \le t) = u_M(t)$ – the distribution of L. Thus, $u \le u_M$.

The minimality of u_m follows form the minimality of $\underline{u}$. In the case $u_0 = 1$ one can also deduce it follows by duality and similar arguments since the operator defined by

$$v \to A^*v(t) = \int_0^t \left[2v(\alpha(t-s)) - v^2(\alpha(t-s))\right] e^{-s}ds, \quad t \ge 0,$$

continues to be a monotone (increasing) operator on the space of non-negative functions v on $[0, \infty)$ bounded by one. □

Remark 8.1. The non-uniqueness in the case $\alpha > 1$ was obtained using the longest path L, finiteness of which is result of the hyper-explosion, however a third solution for the initial data $u_0 = 0$ was obtained in [2]. This solution, as in the case of u_M, represents a probability distribution function convergent to 1 at infinity, and was constructed by initializing the stochastic recursion (8.3) by U_0 according to the Fréchet extreme value distribution $F_\theta(t) = \exp\{-t^{-b}\}, t \ge 0$, with parameter $b = \frac{\ln 2}{\ln \alpha} > 0$ selected to provide the scaling invariance

$$U \overset{\text{dist}}{=} \frac{1}{\alpha} U_1 \vee U_2.$$

Interestingly, if $\alpha > 1$ then (obviously) there is no scaling invariant solution to $V \overset{\text{dist}}{=} \alpha^{-1} V_1 \wedge V_2$. On the other hand, if $\alpha < 1$ and if the corresponding stochastic recursion is initialized by V_0 according to the type III extreme value distribution for the minimum in the case $\alpha < 1$, then it still cannot lead to another solution due to uniqueness.

9 Athreya's solution to (2.1) via Le Jan-Sznitman iterations.

As it was mentioned in Remark 8.1, Athreya in [2] constructed a global solution to (2.1) corresponding to $u_0 = 0$, different from both $\underline{u}(t) = 0$ and $\overline{u}(t) = \mathbb{P}(L \le t)$ via a procedure described in Section 8. Here we exemplify the versatility of the iterative approach introduced in section 4 by recovering this solution using our technique.

Let $G : \mathbb{R} \to [0, \infty)$ be a given (continuous) function. We will initialize the Le Jan-Sznitman iterations by

$$X_0(t) = \begin{cases} 0, & T_\emptyset \ge t, \\ G(t - T_\emptyset), & T_\emptyset < t, \end{cases} \tag{9.1}$$

and iterate $X_n(t)$ as in (4.1). Note that, in contrast to the previous applications where $X_0(t)$ was chosen to be deterministic, in (9.1) $X_0(t)$ is a stochastic process that depends on the root clock $T_\emptyset$.

For $n \in \mathbb{N}$ denote

$$L_n := \max\{\theta_v : \ |v| = n\}, \tag{9.2}$$

representing the time-length of the longest path of n branches in Le Jan-Sznitman tree. (Recall, θ_v is the time-length of a branch v, see (2.3).)

Assume $L_1 (= \max\{\theta_1, \theta_2\}) < t$. Then

$$X_1(t) = G(\alpha(t - \theta_1))G(\alpha(t - \theta_2)),$$

and by induction, if $L_n < t$, then

$$X_n(t) = \prod_{|v|=n} G(\alpha^n (t - \theta_v)). \tag{9.3}$$

Denote

$$F(t) := \mathbb{E}(X_0(t)) = \int_0^t e^{-s} G(t - s)\, \mathrm{d}s. \tag{9.4}$$

Clearly, (9.4) is equivalent to the initial value problem

$$F'(t) + F(t) = G(t), \quad t \ge 0, \qquad F(0) = 0. \tag{9.5}$$

Using (9.4), we can compute the conditional expectations:

$$\mathbb{E}\left(X_{n+1}(t)\, \mathbb{1}_{L_{n+1}<t} \,|\, T_v,\ |v| \le n\right) = \prod_{|v|=n} F^2(\alpha^{n+1}(t - \theta_v)). \tag{9.6}$$

It turns out in the case $u_0 = 0$ multiplication by $\mathbb{1}_{L_{n+1}<t}$ in the equations above can be removed, resulting in the following Lemma.

Lemma 9.1. *Assume*

$$F^2(\alpha t) \le G(t) \qquad \textit{for all } t > 0. \tag{9.7}$$

Than, for any $u_0 \ge 0$ the following super-martingale property holds for $X_n(t)\mathbb{1}_{L_n<t}$:

$$\mathbb{E}\left(X_{n+1}(t)\mathbb{1}_{L_{n+1}<t}\,|\,T_v,\ |v|\le n\right)\le X_n(t)\mathbb{1}_{L_n<t}\quad a.s. \tag{9.8}$$

Moreover, the case $u_0=0$

$$X_n(t)=X_n(t)\,\mathbb{1}_{L_n<t}=\prod_{|v|=n}G(\alpha^n(t-\theta_v))\quad a.s, \tag{9.9}$$

and thus $X_n(t)$ itself is a non-negative super-martingale, i.e.

$$\mathbb{E}\left(X_{n+1}(t)\,|\,T_v,\ |v|\le n\right)\le X_n(t)\quad a.s. \tag{9.10}$$

Proof. The property (9.8) follows immediately from (9.6) and (9.7).

In the case $u_0=0$, if $L_n>t$ then $X_n(t)=0$ (regardless of the initialization X_0) and thus (noting that $\mathbb{P}(L_n=t)=0$)

$$X_n(t)=X_n(t)\,\mathbb{1}_{L_n\le t},\quad \text{a.s},$$

and so, keeping in mind that $G(t)=0$ and $F(t)=0$ for $t\le 0$, (9.9) and 9.10) hold.

□

Proposition 9.1. *Let $u_0=0$ and $X_n(t)$ be the iterated system of stochastic processes (4.1) with the initialization (9.1), with G and F satisfying (9.7), F as in (9.4) .*

Then there exists a stochastic process $X_G(t)$ such that

$$X_G(t)=\lim_{n\to\infty}X_n(t)\qquad a.s.$$

Moreover, if for any $t\ge 0$ the processes $\{X_n(t)\}$ are uniformly integrable, then the convergence above holds in L^1 and

$$u_G(t):=\mathbb{E}(X_G(t))=\lim_{n\to\infty}\mathbb{E}(X_n)\quad \text{is a global in time solution to (2.2)}$$

satisfying

$$u_G(t)\le F(t)\qquad \text{for all } t\ge 0. \tag{9.11}$$

Proof. By Lemma 9.1 $X_n(t)$ is a non-negative super-martingale, and thus it is convergent as $n\to\infty$. Moreover, by Doob's martingale convergence theorem, if $X_n(t)$ are uniformly integrable, then the convergence holds in L^1, and therefore $\mathbb{E}(X_n(t))\to\mathbb{E}(X_G(t))$. To prove that $u_G(t)$ is a solution note that $\mathbb{E}(X_0(t))=F(t)$, and by induction, assuming $\mathbb{E}(X_n(t))\le F(t)$, we obtain, using (4.1)

$$\mathbb{E}(X_{n+1}(t))=\int_0^t e^{-s}\,\mathbb{E}^2\left(X_{n-1}(\alpha(t-s))\right)\,\mathrm{d}s\le\int_0^t e^{-s}\,F^2(\alpha(t-s))\,\mathrm{d}s. \tag{9.12}$$

Applying (9.7):

$$\mathbb{E}(X_{n+1}(t))\le\int_0^t e^{-s}\,G(t-s)\,\mathrm{d}s=F(t).$$

Thus, $\mathbb{E}(X_n(t))\le F(t)$ for all n and t. Thus the estimate (9.11) holds and using Lebesgue dominated convergence theorem to pass to the limit in (9.12) we conclude that $u_G(t)$ is indeed a solution to (2.2) corresponding to $u_0=0$.

□

Remark 9.1. If $G(t)\le 1$ for all $t\ge 0$, then the uniform integrability condition in Proposition 9.1 clearly holds, and so the scheme above generates a solution. In particular, if $G(t)=1$, $t\ge 0$, then $F(t)=1-e^{-t}$, and all the assumption of Proposition 9.1 are satisfied. Thus we re-discover the solution studied in Section 8: $\overline{u}(t)=\mathbb{P}(L<t)$.

Athreya's solution.

To obtain the above-mentioned Athreya's solution, set $F(t) = F_A(t)$, where

$$F_A(t) = e^{-t^{-b}}, \qquad t \geq 0,\ b = \frac{\ln 2}{\ln \alpha}.$$

(i.e. F_A is a CDF of a Frechét distribution). This choice of parameter induces scaling $F_A^2(\alpha t) = F_A(t)$, which implies

$$G_A(t) = F_A(t) + F_A'(t) \geq F(t) = F_A^2(\alpha t),$$

and so condition (9.7) holds. Explicitly,

$$G_A(t) = e^{-t^{-b}}(1 + bt^{-(b+1)}), \qquad t \geq 0,$$

Remark 9.2. An elementary analysis shows that $G_A(t) < 1$ when $b \leq 1$ (i.e, when $\alpha \geq 2$), and as noted in Remark 9.1, the uniform integrability assumption of Proposition 9.1 holds. When $b > 1$ (i.e. $1 < \alpha < 2$), G_A admits a global maximum at $t = 1$, $G_A(1) = e^{-1}(1 + b)$ (see Lemma 9.2). Notice that $G_A(1) > 1$ when $b > e - 1$ (i.e, $1 < \alpha < 2^{1/(e-1)}$), and the uniform integrability is less straightforward.

The following elementary facts about F_A and G_A will be useful in subsequent analysis. We focus on the case $b > e - 1$ (see fig. 3), but of course similar analysis can be done for $b \leq e - 1$.

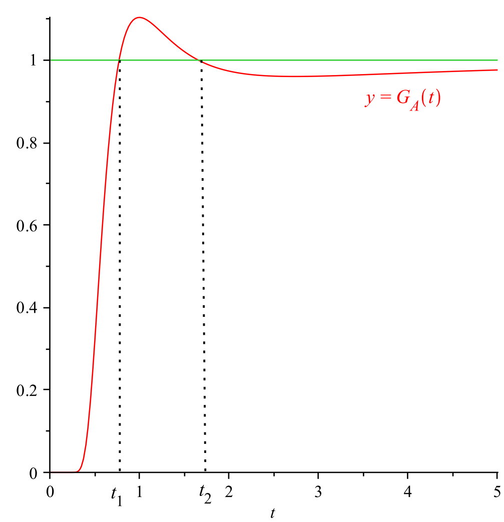

Figure 3: The graph of $G_A(t)$ in the case $b = 2$.

Lemma 9.2. *Let* $b = \frac{\ln 2}{\ln \alpha} > e - 1$. *Then*

1. $G_A(t)$ *has a unique global maximum at* $t_M = 1$ *and a unique local minimum at a* $t_m > b$, *with* $G_A(1) = e^{-1}(1 + b) > 1$, $G_A(t_m) < 1$, *and* $\lim_{t\to\infty} G_A(t) = 1$;
2. *Consequently there exist* t_1, t_2, $0 < t_1 < 1 < t_2 < t_m$, *such that* $G_A(t) > 1$ *if and only if* $t \in (t_1, t_2)$;
3. $e^{-t} = o(1 - F_A(t))$ *as* $t \to \infty$ *and so there exists a* $t_0 > 0$ *such that* $e^{-t} \leq 1 - F_A(t)$ *for all* $t \geq t_0$,

where t_m, t_0, t_1, t_2 *depend only on* α.

Proof. Part 3 follows easily, e.g. using L'Hospital rule, and Part 2 clearly follows from Part 1. To prove Part 1 compute

$$G'_A(t) = e^{-t^{-b}}\, bt^{2(b+1)} \left(t^{b+1} - (b+1)t^b + b\right).$$

One of the obvious critical point is $t_M = 1$, to find the other, note the $G'_A(t)$ has the same sign as

$$h(t) = t^{b+1} - (b+1)t^b + b.$$

Clearly, $h(0) = b$ and $h(t) \to \infty$ as $t \to \infty$. Also,

$$h'(t) = (b+1)t^b - b(b+1)t^{b-1} = (b+1)\, t^b \left(1 - \frac{b}{t}\right),$$

so the unique (global) minimum of h is attained at $t = b$, $h(b) = b - b^b < 0$, and since h is positive at 0 and infinity, there are exactly two zeroes for h – the critical points for G_A: $0 < t_M < b < t_m$, where $t_M(= 1)$ is a local maximum and t_m is a local minimum. So we have $G_A(t_M) = G_A(1) > 1$ and, since $G_A(t) \to 1$ as $t \to \infty$, we must have $G_A(t_2) < 1$ and $G_A(t) \le 1$ for $t > t_2$. Thus, Part 1 of the Lemma holds. ☐

Claim 2. *The iterative sequence $X_n[A](t)$ that corresponds to $G_A(t)$ is uniformly integrable.*

Proof. By Remark 9.2, we know that $\{X_n[A](t)\}$ is uniformly integrable when $\alpha \ge \alpha_0 = 2^{1/(e-1)}$ (equivalently $b \ge 1 - e$).

We will show that for any $\alpha \in (1, \alpha_0)$ there exists $\gamma = \gamma(\alpha) > 1$ such that

$$I(t) := \int_0^t e^{-s}\, G_A^\gamma(t-s)\mathrm{d}s \le 1 \qquad \text{for all } t \ge 0. \tag{9.13}$$

Assuming (9.13) we get

$$\mathbb{E}(X_0^\gamma[A](t)) = \int_0^t e^{-s}\, G_A^\gamma(t-s)\mathrm{d}s \le 1$$

and by induction, if $\mathbb{E}(X_n^\gamma[A](t)) \le 1$ then by (4.1) with $u_0 = 0$ we obtain

$$\mathbb{E}(X_{n+1}^\gamma[A](t)) \le \int_0^t e^{-s}\, \left(\mathbb{E}(X_n^\gamma[A](\alpha(t-s))\right)^2\, \mathrm{d}s \le \int_0^t e^{-s}\, 1\, \mathrm{d}s \le 1.$$

Therefore, $\mathbb{E}(X_n^\gamma[A](t)) \le 1$ for all n (and all t), which, by de la Valeé-Poussin theorem, implies $\{X_n^\gamma[A](t)\}$ is uniformly integrable.

Thus, to complete the proof we need to show (9.13).

Using Lemma 9.2, for $t > t_2$ write

$$\begin{aligned} I(t) &= e^{-t} \int_0^t e^s\, G_A^\gamma(s)\mathrm{d}s = e^{-t} \left(\int_0^{t_1} e^s\, G_A^\gamma(s)\mathrm{d}s + \int_{t_1}^{t_2} e^s\, G_A^\gamma(s)\mathrm{d}s + \int_{t_2}^{t} e^s\, G_A^\gamma(s)\mathrm{d}s \right) \\ &= e^{-t}(I_1(t_1) + I_2(t_2) + I_3(t)). \end{aligned}$$

Note that

$$I_1(t) \le \int_0^t e^s\, G_A(s)\mathrm{d}s, \qquad 0 \le t \le t_1, \tag{9.14}$$

$$I_3(t) \le \int_{t_2}^{t} e^s\, G_A(s)\mathrm{d}s, \quad t > t_2, \tag{9.15}$$

and

$$I_2(t) \le \int_{t_1}^{t} e^s\, G_A(s)\mathrm{d}s + \int_{t_1}^{t} e^s\, (G_A^{\gamma}(s) - G_A(s))\mathrm{d}s, \qquad t_1 < t \le t_2.$$

Consider functions $\phi(x) = x^{\gamma} - x$, $x \in [1, x_M]$, where $x_M = G_A(1)$ – the global maximum of G_A. Elementary calculus shows that $\phi(x) \le \phi(x_M)$ and for any $\delta > 0$ we can choose $\gamma = \gamma(\delta, \alpha) > 1$ such that $\phi(x_M) < \delta$. In particular, using t_0 from Lemma 9.2, to set

$$\delta = \frac{1 - F_A(t_0)}{e^{t_2} - e^{t_1}},$$

we get, for appropriate $\gamma > 1$ (depending only on α, since δ above depends only on α),

$$I_2(t) \le \int_{t_1}^{t} e^s\, G_A(s)\mathrm{d}s + \int_{t_1}^{t} e^s\, \frac{1 - F_A(t_0)}{e^{t_2} - e^{t_1}}\, \mathrm{d}s \le \int_{t_1}^{t} e^s\, G_A(s)\mathrm{d}s + (1 - F_A(t_0))\frac{e^t - e^{t_1}}{e^{t_2} - e^{t_1}}\,. \tag{9.16}$$

Using (9.14)-(9.16) we obtain that for all $t > 0$

$$I(t) \le e^{-t} \int_{0}^{t} e^s\, G_A(s)\, \mathrm{d}s + (1 - F_A(t_0))\, e^{-t} = F_A(t) + (1 - F_A(t_0))\, e^{-t}. \tag{9.17}$$

When $0 \le t \le t_0$ (9.17) yields

$$I(t) \le F_A(t) + 1 - F_A(t_0) \le 1, \qquad \text{since } F_A \text{ is increasing.}$$

When $t > t_0$, we have $e^{-t} < (1 - F_A(t))$, and therefore from (9.17) it follows that

$$I(t) \le F_A(t) + e^{-t} \le F_A(t) + (1 - F_A(t)) = 1.$$

Thus for this choice of γ we have $I(t) \le 1$ for all t, i.e. (9.13) holds.

□

Since for any $\alpha > 1$, the corresponding iterative sequence $X_n[A](t)$ is uniformly integrable, we can apply Proposition 9.1 to obtain a solution $u_A(t) = \mathbb{E}(X_A(t))$, where $X_A(t)$ is the limit of $X_n[A](t)$. This solution is precisely the one obtained by Athreya in [2]. Indeed, one can verify that the fixed point CDF iterations for (8.3) starting with the CDF given by $F_A(t)$ – the method employed in [2] – are *exactly the same* as for the expected values $\mathbb{E}(X_n[A])$, namely (9.12) (Recall, $\mathbb{E}(X_0[A](t)) = F_A(t)$).

10 Acknowledgments

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